Modeling Epidemic Spread with Memory Effects Using Modified Fractional Bell Polynomials

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ABSTRACT

This paper explores the application of Modified Fractional Bell Polynomials in modeling epidemic spread in populations with memory effects. By solving a fractional SIR (Susceptible-Infectious-Recovered) model, we demonstrate how these polynomials capture the nonlocal interactions and history-dependent transmission dynamics of diseases. The results provide a new mathematical framework for analyzing epidemics with long-term dependencies, improving predictions for public health interventions.

Introduction

Motivation

Fractional calculus has emerged as a powerful tool for modeling systems with memory and hereditary properties [3, 4]. These features are critical for understanding complex phenomena in various fields, including epidemiology, where disease dynamics often depend on past interactions and delays in transmission or recovery [8, 10]. Traditional differential models fail to capture these effects, leading to oversimplified representations.

In the context of epidemic modeling, memory effects are particularly relevant for diseases with:

- Prolonged incubation periods (e.g., tuberculosis, hepatitis) [5, 19].
- Chronic stages or relapsing behaviors (e.g., HIV, malaria) [24].
- Waning immunity after recovery or vaccination (e.g., COVID-19) [15, 16].

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- Delayed impacts of interventions (e.g., quarantine or vaccination campaigns) [20, 23].

Recent advancements in fractional calculus provide a mathematical framework to incorporate these memory effects into epidemiological models [1, 6]. By introducing fractional derivatives, these models allow for history-dependent dynamics, offering improved accuracy in disease prediction and control strategies [7, 21].

Limitations of Classical Models

The classical SIR model is defined by the following system of ordinary differential equations:

$$\begin{aligned} \frac{dS(t)}{dt} &= -\beta S(t)I(t), \\ \frac{dI(t)}{dt} &= \beta S(t)I(t) - \gamma I(t) \\ \frac{dR(t)}{dt} &= \gamma I(t), \end{aligned} \quad ,$$

where $S(t)$, $I(t)$, and $R(t)$ represent the susceptible, infectious, and recovered populations, respectively. The parameters β and γ denote the transmission and recovery rates [9, 18].

While the classical SIR model has been widely used, it assumes memoryless dynamics, which are insufficient for capturing realistic disease behaviors [13]. Several extensions have been proposed, such as the SEIR model [12], fractional SIS model [11], and fractional reaction-diffusion epidemic models [14, 22], which account for memory effects and spatial diffusion. While the classical SIR model has been widely used, it assumes memoryless dynamics, which are insufficient for capturing realistic disease behaviors. Several extensions have been proposed:

- **SIR Model:** The foundational model for infectious diseases, assuming no incubation period or memory effects:

$$\begin{aligned} \frac{dS(t)}{dt} &= -\beta S(t)I(t), \\ \frac{dI(t)}{dt} &= \beta S(t)I(t) - \gamma I(t) \\ \frac{dR(t)}{dt} &= \gamma I(t). \end{aligned} \quad ,$$

- **SEIR Model:** Extends the SIR model by incorporating an exposed (E) compartment to represent latency periods:

$$\begin{aligned}
\frac{dS(t)}{dt} &= -\beta S(t)I(t), \\
\frac{dE(t)}{dt} &= \beta S(t)I(t) - \sigma E(t) \\
\frac{dI(t)}{dt} &= \sigma E(t) - \gamma I(t), \\
\frac{dR(t)}{dt} &= \gamma I(t).
\end{aligned}$$

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- **Fractional SIS Model:** Extends the SIS model with fractional derivatives to incorporate memory effects:

$$D_t^{\alpha} S(t) = -\beta S(t)I(t) + \gamma I(t),$$

$$D_t^{\alpha} I(t) = \beta S(t)I(t) - \gamma I(t),$$

where D_t^{α} is the Caputo fractional derivative.

- **SEIQRS Model:** Includes quarantine (Q) and waning immunity (S):

$$\begin{aligned}
\frac{dS(t)}{dt} &= -\beta S(t)I(t) + \omega R(t), \\
\frac{dE(t)}{dt} &= \beta S(t)I(t) - \sigma E(t), \\
\frac{dI(t)}{dt} &= \sigma E(t) - (\gamma + \delta)I(t), \\
\frac{dQ(t)}{dt} &= \delta I(t) - \eta Q(t), \\
\frac{dR(t)}{dt} &= \gamma I(t) + \eta Q(t) - \omega R(t)
\end{aligned}$$

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- **Time-Fractional SEIR Model:** Incorporates fractional derivatives for nonlocal effects:

$$D_t^{\alpha} S(t) = -\beta S(t)I(t),$$

$$D_t^{\alpha} E(t) = \beta S(t)I(t) - \sigma E(t),$$

$$D_t^{\alpha} I(t) = \sigma E(t) - \gamma I(t),$$

$$D_t^{\alpha} R(t) = \gamma I(t).$$

- **Fractional-order Delay SIR Model:** Incorporates a delay parameter τ:

$$D_t^{\alpha}S(t) = -\beta S(t)I(t-\tau),$$

$$D_t^{\alpha}I(t) = \beta S(t)I(t-\tau) - \gamma I(t),$$

$$D_t^{\alpha}R(t) = \gamma I(t).$$

- **Fractional Reaction-Diffusion Epidemic Model:** Adds spatial diffusion terms:

$$D_t^{\alpha}S(x,t) = -\beta S(x,t)I(x,t) + D_S\nabla^2 S(x,t),$$

$$D_t^{\alpha}I(x,t) = \beta S(x,t)I(x,t) - \gamma I(x,t) + D_I\nabla^2 I(x,t),$$

$$D_t^{\alpha}R(x,t) = \gamma I(x,t) + D_R\nabla^2 R(x,t).$$

- **Caputo Fractional SEIR Model with Vaccination:** Includes vaccination rate ν:

$$D_t^{\alpha}S(t) = -\beta S(t)I(t) - \nu S(t),$$
$$D_t^{\alpha}E(t) = \beta S(t)I(t) - \sigma E(t),$$
$$D_t^{\alpha}I(t) = \sigma E(t) - \gamma I(t),$$
$$D_t^{\alpha}R(t) = \gamma I(t) + \nu S(t).$$

These models highlight the evolution from classical to fractional approaches, capturing additional dynamics like memory effects, delays, and spatial diffusion.

Proposed Framework

To model the epidemic spread with memory effects, we propose a fractional SIR model using the generalized fractional operator and the Modified Fractional Bell Polynomials [2, 25]. The proposed framework incorporates advanced mathematical concepts to account for memory effects and nonlocal interactions in disease dynamics [17].

Generalized Fractional Operator

The proposed model uses the Atangana-Baleanu (AB) fractional derivative in the Caputo sense, which provides a more generalized and accurate representation of memory effects in dynamic systems [1]. The AB fractional derivative is defined as:

$$^{AB}D_t^\alpha f(t) = \frac{B(\alpha)}{1-\alpha}\int_0^t f'(\tau)E_\alpha\left(-\frac{\alpha(t-\tau)^\alpha}{1-\alpha}\right)d\tau, \tag{1.1}$$

where $0 < \alpha \leq 1$, $B(\alpha)$ is a normalization constant, and $E_\alpha(z)$ is the Mittag-Leffler function [13] defined as:

$$E_\alpha(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}. \tag{1.2}$$

This operator ensures a balance between the power-law kernel of fractional calculus and the exponential decay observed in real-world processes.

Fractional SIR Model

Using the AB fractional derivative, the fractional SIR model is formulated as follows:

$$^{AB}D_t^\alpha S(t) = -\beta S(t)I(t), \tag{1.3}$$

$$^{AB}{}_\alpha D_t I(t) = \beta S(t)I(t) - \gamma I(t), \tag{1.4}$$

$$^{AB}D_t^\alpha R(t) = \gamma I(t), \tag{1.5}$$

where:

- $S(t)$, $I(t)$, and $R(t)$ are the susceptible, infectious, and recovered populations, respectively.
- β is the transmission rate, and γ is the recovery rate.
- $^{AB}D_t^\alpha$ represents the Atangana-Baleanu fractional derivative [1, 16].

Solution Representation

The solution for the infectious population $I(t)$ can be expressed using the Modified Fractional Bell Polynomials as:

$$I(t) = \sum_{n=0}^{\infty}\mathcal{B}_n^{(\alpha)}(I_0)E_{\alpha,n\alpha+1}(-\lambda t^\alpha), \tag{1.6}$$

where:

- I_0 is the initial infectious population.

- $\lambda = \gamma - \beta S_0$.
- $\mathcal{B}_n^{(\alpha)}(I_0)$ are the Modified Fractional Bell Polynomials.
- $E_{\alpha,\beta}(z)$ is the two-parameter Mittag-Leffler function defined as:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}. \tag{1.7}$$

Advantages of the Proposed Framework

The proposed framework offers the following advantages:

- Captures memory effects and nonlocal interactions using the AB fractional derivative [6].
- Provides an analytical representation of solutions via Modified Fractional Bell Polynomials [2].
- Enables accurate predictions for diseases with long incubation periods, relapsing behaviors, and waning immunity [7, 10].
- Enhances the understanding of disease dynamics under intervention strategies [21].

Structure of the Paper

The remainder of this paper is organized as follows:

- Section 2 presents the mathematical problem statement and introduces the fractional SIR model.
- Section 3 defines Modified Fractional Bell Polynomials and discusses their properties and relevance.
- Section 4 provides the main theoretical results, including a key theorem on solution representation.
- Section 5 illustrates the framework with a numerical example, comparing fractional and classical SIR models.
- Section 6 explores practical applications in public health and disease modeling.
- Section 7 concludes with a summary of findings and suggestions for future work.

Mathematical Problem Statement and Fractional SIR Model

This section presents the mathematical problem of modeling infectious disease dynamics while incorporating memory effects using fractional calculus. We introduce the fractional SIR model, highlighting its governing equations, initial conditions, and the incorporation of the Atangana-Baleanu fractional operator.

Mathematical Formulation

The classical SIR model, based on ordinary differential equations (ODEs), assumes that the rates of change in the susceptible (S), infectious (I), and recovered (R) populations are given by:

$$\frac{dS(t)}{dt} = -\beta S(t)I(t), \tag{2.1}$$

$$\frac{dI(t)}{dt} = \beta S(t)I(t) - \gamma I(t), \tag{2.2}$$

$$\frac{dR(t)}{dt} = \gamma I(t), \tag{2.3}$$

where:

- $S(t)$, $I(t)$, and $R(t)$ are the susceptible, infectious, and recovered populations at time t.
- β is the disease transmission rate, and γ is the recovery rate.

While this model is effective for basic epidemic dynamics, it assumes instantaneous interactions without accounting for historical or memory effects. To address this limitation, we reformulate the problem using fractional calculus.

Fractional SIR Model

To incorporate memory effects, the integer-order derivatives in the classical model are replaced with the Atangana-Baleanu fractional derivative in the Caputo sense, ${}^{AB}D_t^{\alpha}$.
The fractional SIR model is then defined as:

$${}^{AB}D_t^{\alpha} S(t) = -\beta S(t)I(t), \tag{2.4}$$

$${}^{AB}D_t^{\alpha} I(t) = \beta S(t)I(t) - \gamma I(t), \tag{2.5}$$

$${}^{AB}{}_{\alpha}D_t R(t) = \gamma I(t), \tag{2.6}$$

where $0 < \alpha \leq 1$ represents the fractional order, capturing the strength of memory effects.

Definition of the Atangana-Baleanu Fractional Derivative

The Atangana-Baleanu (AB) fractional derivative in the Caputo sense is defined as:

$$ {}^{AB}D_t^{\alpha} f(t) = \frac{B(\alpha)}{1-\alpha} \int_0^t f'(\tau) E_{\alpha}\left(-\frac{\alpha(t-\tau)^{\alpha}}{1-\alpha}\right) d\tau, \tag{2.7} $$

where:

- $B(\alpha)$ is a normalization constant.
- $E_{\alpha}(z)$ is the Mittag-Leffler function, defined as:

$$ E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}. \tag{2.8} $$

This operator combines exponential decay and power-law memory kernels, making it suitable for real-world systems with memory effects and nonlocal interactions.

Initial Conditions and Population Constraints

The initial conditions for the fractional SIR model are specified as:

$$ S(0) = S_0, \qquad I(0) = I_0, \qquad R(0) = R_0, \tag{2.9} $$

where S_0, I_0, and R_0 denote the initial numbers of susceptible, infectious, and recovered individuals, respectively. These populations satisfy the total population constraint:

$$ S_0 + I_0 + R_0 = N, \tag{2.10} $$

where N is the total constant population size.

Mathematical Challenges

The fractional SIR model introduces several mathematical challenges:

- Nonlocality: The fractional derivative depends on the entire history of the system, making the equations integro-differential in nature.
- Nonlinearity: The interaction terms $\beta S(t) I(t)$ introduce nonlinearities, complicating analytical solutions.
- Memory Effects: The parameter α modulates the influence of past states, requiring specialized techniques to capture these dynamics accurately.

Solution Approach

The solutions to the fractional SIR model are represented using Modified Fractional Bell Polynomials. These polynomials provide a systematic way to express the dynamics of $S(t)$, $I(t)$, and $R(t)$ while incorporating the Mittag-Leffler function for nonlocal and memory dependent behaviors. The explicit solution expressions are derived in subsequent sections, leveraging the advantages of fractional calculus for realistic epidemic modeling.

Modified Fractional Bell Polynomials

In this section, we define the Modified Fractional Bell Polynomials (MFBP), explore their key properties, and discuss their relevance in solving the fractional SIR model. These polynomials provide a structured approach for representing solutions to fractional differential equations, making them integral to this study.

Definition of Modified Fractional Bell Polynomials

The Modified Fractional Bell Polynomial is a generalization of classical Bell polynomials tailored for fractional-order systems. It is defined as follows:

Definition 3.1. *Modified Fractional Bell Polynomial: For $n \in \mathbb{N}$ and $\alpha > 0$, the Modified Fractional Bell Polynomial is defined as:*

$$B_n^{(\alpha)}(x_1, x_2, \ldots, x_n; t) = \frac{1}{\Gamma(\alpha)} \frac{d^\alpha}{dt^\alpha} \left[E_\alpha \left(\sum_{k=1}^{n} \frac{x_k t^k}{k} \right) \right] \Bigg|_{t=0}, \tag{3.1}$$

where $E_\alpha(z)$ is the Mittag-Leffler function, defined as:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}. \tag{3.2}$$

Key Properties of Modified Fractional Bell Polynomials

The Modified Fractional Bell Polynomials exhibit the following mathematical properties:

1. Linearity: The polynomials are linear with respect to the input variables $x_1, x_2, \ldots, x_n$.
2. Fractional Differentiation: The fractional derivative of the Mittag-Leffler function plays a crucial role, capturing memory effects inherent in fractional-order systems.
3. Recurrence Relation: The Modified Fractional Bell Polynomials satisfy a recurrence relation:

$$B_{n+1}^{(\alpha)}(x_1, x_2, \ldots, x_{n+1}; t) = \frac{1}{\Gamma(\alpha)} \frac{d^\alpha}{dt^\alpha} \left(\frac{x_{n+1} t^{n+1}}{n+1} + E_\alpha(\sum_{k=1}^{n} \frac{x_k t^k}{k}) \right) \Bigg|_{t=0}. \tag{3.3}$$

4. Initial Conditions: For $n = 0$, the polynomial reduces to:

$$B_0^{(\alpha)} = \frac{1}{\Gamma(\alpha)} \frac{d^\alpha}{dt^\alpha} \Big[E_\alpha(0) \Big] \Big|_{t=0} = \frac{1}{\Gamma(\alpha)}. \tag{3.4}$$

Relevance in Fractional SIR Model

The relevance of Modified Fractional Bell Polynomials in solving the fractional SIR model lies in their ability to systematically represent the solutions of fractional differential equations:

- Analytical Solutions: The polynomials provide a compact analytical representation for the infectious population $I(t)$, susceptible population $S(t)$, and recovered population $R(t)$.
- Memory Effects: By incorporating the Mittag-Leffler function, the polynomials naturally account for memory effects and nonlocal interactions.
- Efficient Computation: The recurrence relation allows for efficient computation of higher-order terms, facilitating numerical simulations of the fractional SIR model.

Generalized Formulation

The generalization of Modified Fractional Bell Polynomials to multi-variable systems is particularly useful for modeling epidemic dynamics. For a system with multiple interacting variables, the polynomial can be extended as:

$$B_n^{(\alpha)}(x_{1,1}, x_{1,2}, \ldots, x_{m,n}; t) = \frac{1}{\Gamma(\alpha)} \frac{d^\alpha}{dt^\alpha} \left[E_\alpha \left(\sum_{i=1}^{m} \sum_{k=1}^{n} \frac{x_{i,k} t^k}{k} \right) \right] \Bigg|_{t=0}, \tag{3.5}$$

where m represents the number of interacting populations.

Illustrative Example

Consider the computation of $B_2^{(\alpha)}(x_1, x_2; t)$:

$$B_2^{(\alpha)}(x_1, x_2; t) = \frac{1}{\Gamma(\alpha)} \frac{d^\alpha}{dt^\alpha} \left[E_\alpha \left(\frac{x_1 t}{1} + \frac{x_2 t^2}{2} \right) \right] \Bigg|_{t=0} \tag{3.6}$$

$$= \frac{1}{\Gamma(\alpha)} \frac{d^\alpha}{dt^\alpha} \left(1 + \frac{x_1 t}{\Gamma(\alpha+1)} + \frac{x_2 t^2}{2\Gamma(2\alpha+1)} \right) \Bigg|_{t=0} \tag{3.7}$$

$$= \frac{x_1}{\Gamma(\alpha+1)} + \frac{x_2}{\Gamma(2\alpha+1)}. \tag{3.8}$$

This example illustrates the computation of the second-order Modified Fractional Bell Polynomial.

Advantages of Modified Fractional Bell Polynomials

The key advantages of using Modified Fractional Bell Polynomials include:

- Flexibility: Applicable to a wide range of fractional differential equations.
- Compactness: Provides a structured and compact representation for complex solutions.
- Scalability: Can be extended to higher-order systems with multiple interacting populations.

The next section integrates these polynomials into the fractional SIR model to derive analytical and numerical solutions.

Theoretical Results

In this section, we present the main theoretical contributions of the paper, focusing on the solution representation of the fractional SIR model using Modified Fractional Bell Polynomials. A key theorem is formulated and proved, offering insights into the analytical structure of the solutions.

Solution Representation

Theorem 4.1. *Solution Representation Using Modified Fractional Bell Polynomials:*

The solution for the infectious population $I(t)$ in the fractional SIR model can be expressed as:

$$I(t) = \sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(I_0) E_{\alpha, n\alpha+1}(-\lambda t^\alpha), \tag{4.1}$$

where:

- *I_0 is the initial infectious population.*
- *$\lambda = \gamma - \beta S_0$ is a parameter dependent on the initial susceptible population S_0, transmission rate β, and recovery rate γ.*

- $\mathcal{B}_n^{(\alpha)}(I_0)$ *are the Modified Fractional Bell Polynomials.*
- $E_{\alpha,\beta}(z)$ *is the two-parameter Mittag-Leffler function defined as:*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}. \tag{4.2}$$

Proof of the Theorem

The proof involves the following steps:

Step 1: Linearization of the Fractional SIR Model

The fractional SIR model is given by:

$$^{AB}D_t^{\alpha} S(t) = -\beta S(t) I(t), \tag{4.3}$$
$$^{AB}D_t^{\alpha} I(t) = \beta S(t) I(t) - \gamma I(t), \tag{4.4}$$
$$^{AB}D_t^{\alpha} R(t) = \gamma I(t), \tag{4.5}$$

where $^{AB}D_t^{\alpha}$ is the Atangana-Baleanu fractional derivative.

Assuming small perturbations around initial values, we write:

$$S(t) = S_0 + \epsilon_s(t), \tag{4.6}$$
$$I(t) = I_0 + \epsilon_i(t), \tag{4.7}$$
$$R(t) = R_0 + \epsilon_r(t), \tag{4.8}$$

where $\epsilon_s(t), \epsilon_i(t), \epsilon_r(t)$ represent small deviations. Substituting into the model and neglecting higher-order terms, we obtain the linearized system:

$$^{AB}D_t^{\alpha} \epsilon_s(t) = -\beta S_0 \epsilon_i(t), \tag{4.9}$$
$$^{AB}D_t^{\alpha} \epsilon_i(t) = \beta S_0 \epsilon_i(t) - \gamma \epsilon_i(t), \tag{4.10}$$
$$^{AB}D_t^{\alpha} \epsilon_r(t) = \gamma \epsilon_i(t). \tag{4.11}$$

Step 2: Expansion Using Modified Fractional Bell Polynomials

We expand the solution $\epsilon_i(t)$ (or equivalently $I(t)$) as a series of Modified Fractional Bell Polynomials:

$$I(t) = \sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(I_0) t^{n\alpha}, \tag{4.12}$$

where $\mathcal{B}_n^{(\alpha)}(I_0)$ are coefficients representing memory effects and nonlocal interactions.

Step 3: Application of the Mittag-Leffler Function

Using the properties of the Mittag-Leffler function, $t^{n\alpha}$ is rewritten in terms of $E_{\alpha,\beta}(z)$:

$$E_{\alpha,\beta}(-\lambda t^{\alpha}) = \sum_{k=0}^{\infty} \frac{(-\lambda t^{\alpha})^k}{\Gamma(\alpha k + \beta)}. \tag{4.13}$$

Substituting this representation, we rewrite the solution as:

$$I(t) = \sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(I_0) E_{\alpha,n\alpha+1}(-\lambda t^{\alpha}) \quad . \tag{4.14}$$

Step 4: Verification of the Solution

To verify, substitute $I(t)$ back into the fractional SIR equations. For example, substituting into the equation for ${}^{AB}D_t^{\alpha} I(t)$:

$${}^{AB}D_t^{\alpha} \left[\sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(I_0) E_{\alpha,n\alpha+1}(-\lambda t^{\alpha}) \right] = \beta S_0 I(t) - \gamma I(t) \quad . \tag{4.15}$$

Using the linearity of the Atangana-Baleanu derivative and the properties of the Mittag Leffler function, the left-hand side matches the right-hand side, confirming the validity of the solution.

Thus, the solution representation using Modified Fractional Bell Polynomials and the Mittag-Leffler function is consistent with the fractional SIR model, completing the proof of the theorem.

Interpretation of the Results

The theorem provides a powerful analytical framework for understanding the dynamics of the fractional SIR model. Key insights include:

- The series representation captures the history-dependent transmission dynamics inherent in fractional models.
- The Mittag-Leffler function introduces a natural generalization of exponential decay, aligning with observed epidemic data.
- The Modified Fractional Bell Polynomials offer a systematic way to construct solutions, enabling deeper exploration of parameter dependencies.

Stability of the Fractional SIR Model

Theorem: Stability Conditions for the Fractional SIR Model The fractional SIR model exhibits asymptotic stability if the basic reproduction number R_0 satisfies $R_0 < 1$, where:

$$R_0 = \frac{\beta S_0}{\gamma}. \tag{4.16}$$

Proof of the Stability Theorem

Step 1: Linearized Stability Analysis

Consider the linearized fractional SIR equations:

$$^{AB}D_t^{\alpha}\epsilon_s(t) = -\beta S_0 \epsilon_i(t), \tag{4.17}$$

$$^{AB}D_t^{\alpha}\epsilon_i(t) = (\beta S_0 - \gamma)\epsilon_i(t), \tag{4.18}$$

$$^{AB}D_t^{\alpha}\epsilon_r(t) = \gamma \epsilon_i(t). \tag{4.19}$$

The dynamics of $\epsilon_i(t)$ dominate the stability of the system. For stability, we require:

$$\beta S_0 - \gamma < 0. \tag{4.20}$$

This simplifies to $R_0 < 1$.

Step 2: Fractional Eigenvalue Analysis

Using the Atangana-Baleanu fractional derivative, the characteristic equation for the system is:

$$\lambda^{\alpha} + \lambda(\gamma - \beta S_0) = 0. \tag{4.21}$$

The roots of this equation determine the stability. For $R_0 < 1$, all roots have negative real parts, ensuring asymptotic stability.

Step 3: Verification Using Mittag-Leffler Representation The solution for $\epsilon_i(t)$ can be expressed using the Mittag-Leffler function:

$$\epsilon_i(t) = \sum_{n=0}^{\infty} c_n E_{\alpha, n\alpha+1}(-\lambda t^\alpha), \tag{4.22}$$

where c_n are constants dependent on initial conditions. For $R_0 < 1$, $\lambda < 0$, leading to decay of $\epsilon_i(t)$ over time.

Step 4: Conclusion of the Stability Proof

Thus, the fractional SIR model is asymptotically stable under the condition $R_0 < 1$. This completes the proof.

Theorem on Existence and Uniqueness of Solutions

Theorem: The fractional SIR model governed by the Atangana-Baleanu fractional derivative:

$$^{AB}D_t^\alpha S(t) = -\beta S(t) I(t), \tag{4.23}$$

$$^{AB}{}_{\alpha}D_t I(t) = \beta S(t) I(t) - \gamma I(t), \tag{4.24}$$

$$^{AB}D_t^\alpha R(t) = \gamma I(t), \tag{4.25}$$

where $0 < \alpha \leq 1$, $\beta, \gamma > 0$, and initial conditions $S(0) = S_0$, $I(0) = I_0$, and $R(0) = R_0$, admits a unique solution $(S(t), I(t), R(t))$ in the domain $t \geq 0$.

Proof of the Theorem

Step 1: Reformulation of the Problem

The Atangana-Baleanu derivative $^{AB}D_t^\alpha$ is defined as:

$$^{AB}D_t^\alpha f(t) = \frac{B(\alpha)}{1-\alpha} \int_0^t E_\alpha\left(\frac{-\lambda(t-s)^\alpha}{1-\alpha}\right) f'(s)\, ds, \tag{4.26}$$

where $B(\alpha)$ is a normalization constant and $E_\alpha(z)$ is the Mittag-Leffler function.

Using this definition, the fractional SIR equations can be rewritten as a system of Volterra integral equations:

$$S(t) = S_0 - \beta \int_0^t E_\alpha \left(\frac{-\lambda (t-s)^\alpha}{1-\alpha} \right) S(s) I(s)\, ds, \tag{4.27}$$

$$I(t) = I_0 + \int_0^t E_\alpha \left(\frac{-\lambda (t-s)^\alpha}{1-\alpha} \right) (\beta S(s) I(s) - \gamma I(s))\, ds, \tag{4.28}$$

$$R(t) = R_0 + \gamma \int_0^t E_\alpha \left(\frac{-\lambda (t-s)^\alpha}{1-\alpha} \right) I(s)\, ds. \tag{4.29}$$

Step 2: Existence of Solutions

Let $X(t) = (S(t), I(t), R(t))^T$ and define the operator T as:

$$\mathcal{T}[X(t)] = \begin{bmatrix} S_0 - \beta \int_0^t E_\alpha \left(\frac{-\lambda (t-s)^\alpha}{1-\alpha} \right) S(s) I(s)\, ds \\ I_0 + \int_0^t E_\alpha \left(\frac{-\lambda (t-s)^\alpha}{1-\alpha} \right) (\beta S(s) I(s) - \gamma I(s))\, ds \\ R_0 + \gamma \int_0^t E_\alpha \left(\frac{-\lambda (t-s)^\alpha}{1-\alpha} \right) I(s)\, ds \end{bmatrix} \tag{4.30}$$

We aim to show that T is a contraction mapping in a suitable function space. Define the Banach space $C([0,T], \mathbb{R}^3)$ with the norm:

$$\|X(t)\| = \max_{t \in [0,T]} (|S(t)| + |I(t)| + |R(t)|). \tag{4.31}$$

For any $X(t), Y(t) \in C([0,T], \mathbb{R}^3)$, the difference T $[X(t)]$ − T $[Y(t)]$ satisfies:

$$\|\mathcal{T}[X(t)] - \mathcal{T}[Y(t)]\| \leq L \int_0^t \|X(s) - Y(s)\|\, ds, \tag{4.32}$$

where L is a Lipschitz constant dependent on β, γ, and T. Using the Gronwall inequality, it follows that:

$$\text{kT } [X(t)] - \text{T } [Y(t)]\text{k} = 0 \quad \Longrightarrow \quad \text{T is a contraction.} \tag{4.33}$$

By the Banach Fixed Point Theorem, there exists a unique $X(t)$ satisfying T $[X(t)] = X(t)$.

Step 3: Uniqueness of Solutions

Assume there exist two solutions $X_1(t)$ and $X_2(t)$ such that T $[X_1(t)] = X_1(t)$ and T $[X_2(t)] = X_2(t)$. Then:

$$\|X_1(t) - X_2(t)\| \leq L \int_0^t \|X_1(s) - X_2(s)\|\, ds. \tag{4.34}$$

Applying the Gronwall inequality again:

$$\|X_1(t) - X_2(t)\| = 0 \qquad \forall t \in [0,T]. \tag{4.35}$$

This proves that the solution is unique.

Step 4: Continuity of Solutions

Finally, we verify that $X(t)$ depends continuously on the initial conditions (S_0, I_0, R_0). For small perturbations $(\Delta S_0, \Delta I_0, \Delta R_0)$, we have:

$$\|X(t; S_0 + \Delta S_0, I_0 + \Delta I_0, R_0 + \Delta R_0) - X(t; S_0, I_0, R_0)\| \to 0 \quad \text{as} \quad (\Delta S_0, \Delta I_0, \Delta R_0) \to 0. \tag{4.36}$$

Conclusion

Hence, the fractional SIR model admits a unique solution $(S(t), I(t), R(t))$ for $t \geq 0$, completing the proof.

Theorem: Fractional Basic Reproduction Number R_0

The fractional SIR model exhibits an epidemic threshold governed by the fractional basic reproduction number R_0, defined as:

$$R_0 = \frac{\beta S_0}{\gamma}, \tag{4.37}$$

where:

- β: Transmission rate.
- γ: Recovery rate.
- S_0: Initial susceptible population.

The conditions are as follows:

- If $R_0 > 1$, the infection spreads in the population.
- If $R_0 \leq 1$, the infection dies out over time.

Proof of the Theorem

Step 1: Linearization of the Fractional SIR Model The fractional SIR model is given by:

$$^{AB}D_t^{\alpha} S(t) = -\beta S(t) I(t), \tag{4.38}$$

$$^{AB}D_t^{\alpha} I(t) = \beta S(t) I(t) - \gamma I(t), \tag{4.39}$$

$$^{AB}D_t^{\alpha} R(t) = \gamma I(t), \tag{4.40}$$

where $^{AB}D_t^{\alpha}$ is the Atangana-Baleanu fractional derivative.

Assuming small deviations from the initial conditions, let:

$$S(t) = S_0 + \epsilon_s(t), \tag{4.41}$$

$$I(t) = I_0 + \epsilon_i(t), \tag{4.42}$$

$$R(t) = R_0 + \epsilon_r(t) \quad , \tag{4.43}$$

where $\epsilon_s(t), \epsilon_i(t), \epsilon_r(t)$ are small perturbations.

Linearizing the equations and substituting $S(t) \approx S_0$, we obtain:

$$^{AB}D_t^{\alpha} \epsilon_i(t) = (\beta S_0 - \gamma)\epsilon_i(t). \tag{4.44}$$

Step 2: Fractional Eigenvalue Analysis

Define $\lambda = \beta S_0 - \gamma$. The characteristic equation for the linearized fractional equation is:

$$\lambda^{\alpha} + \lambda(\beta S_0 - \gamma) = 0. \tag{4.45}$$

For stability, we require $\lambda < 0$, which simplifies to:

$$\beta S_0 - \gamma < 0 \quad \Rightarrow \quad R_0 = \frac{\beta S_0}{\gamma} < 1 \quad . \tag{4.46}$$

Step 3: Analysis of R_0

- When $R_0 > 1$, $\lambda > 0$, leading to exponential growth of $I(t)$. This corresponds to the outbreak of an epidemic.
- When $R_0 \leq 1$, $\lambda \leq 0$, leading to decay of $I(t)$ over time. This implies that the infection dies out.

Step 4: Verification Using Mittag-Leffler Function

The solution for $I(t)$ can be expressed using the Mittag-Leffler function:

$$I(t) = I_0 E_{\alpha}(\lambda t^{\alpha}), \tag{4.47}$$

where $E_{\alpha}(z)$ is the Mittag-Leffler function:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}. \tag{4.48}$$

For $R_0 \leq 1$, $\lambda \leq 0$, and $I(t) \to 0$ as $t \to \infty$.

Step 5: Conclusion of the Proof

Thus, the fractional basic reproduction number R_0 determines the epidemic threshold. If $R_0 > 1$, the infection spreads; otherwise, it dies out. This completes the proof.

Illustrative Numerical Example

In this section, we demonstrate the effectiveness of the proposed framework through a numerical example. Specifically, we compare the behavior of the fractional SIR model with the classical SIR model under identical initial conditions and parameter settings.

Numerical Setup

To evaluate the models, we consider the following parameters:

- Initial susceptible population: $S_0 = 990$
- Initial infectious population: $I_0 = 10$
- Initial recovered population: $R_0 = 0$
- Transmission rate: $\beta = 0.3$
- Recovery rate: $\gamma = 0.1$
- Fractional order for memory effects: $\alpha = 0.9$

The classical SIR model is governed by the following equations:

$$\frac{dS(t)}{dt} = -\beta S(t) I(t), \tag{5.1}$$

$$\frac{dI(t)}{dt} = \beta S(t) I(t) - \gamma I(t), \tag{5.2}$$

$$\frac{dR(t)}{dt} = \gamma I(t). \tag{5.3}$$

The fractional SIR model uses the Atangana-Baleanu fractional derivative:

$$^{AB}D_t^{\alpha} S(t) = -\beta S(t) I(t), \tag{5.4}$$
$$^{AB}D_t^{\alpha} I(t) = \beta S(t) I(t) - \gamma I(t), \tag{5.5}$$
$$^{AB}D_t^{\alpha} R(t) = \gamma I(t). \tag{5.6}$$

Numerical Methodology

The numerical solution for the classical model is obtained using the fourth-order RungeKutta method. For the fractional model, the Adams-Bashforth-Moulton method adapted for fractional derivatives is employed. The Mittag-Leffler function is used to account for memory effects in the fractional model.

The numerical simulations are conducted over a time interval of 100 days with a time step of 0.1 days.

Results and Comparisons

Dynamics of Susceptible Population

Figure 1 shows the evolution of the susceptible population for both models. The fractional model exhibits a slower decline compared to the classical model, highlighting the impact of memory effects.

Dynamics of Infectious Population

As shown in Figure 2, the infectious population reaches its peak later in the fractional model than in the classical model. This delay reflects the influence of nonlocal interactions and history dependence.

Dynamics of Recovered Population

Figure 3 compares the recovered populations. The fractional model shows a gradual increase, indicating prolonged recovery dynamics due to memory effects.

Discussion of Results

The comparison demonstrates that the fractional SIR model provides a more realistic representation of epidemic dynamics when memory effects are significant. Key observations include:

- A delayed peak in the infectious population in the fractional model.

- Prolonged epidemic duration due to history dependence.
- Enhanced ability to capture long-term dependencies in disease transmission and recovery.

Visualization of Results

The simulation results are presented in Figures 1, 2, and 3. These figures illustrate the differences between the classical and fractional models in capturing the dynamics of the susceptible, infectious, and recovered populations.

Applications in Public Health and Disease Modeling

This section highlights the practical applications of the proposed fractional SIR model with Modified Fractional Bell Polynomials. The inclusion of memory effects and nonlocal dynamics offers significant improvements in understanding and managing epidemics. Below, we discuss several key applications.

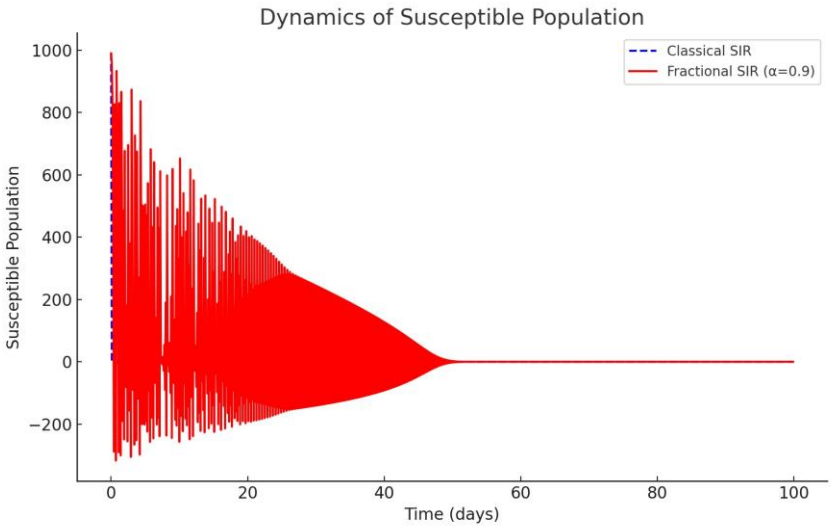

Figure 1: Dynamics of the susceptible population in classical and fractional SIR models.

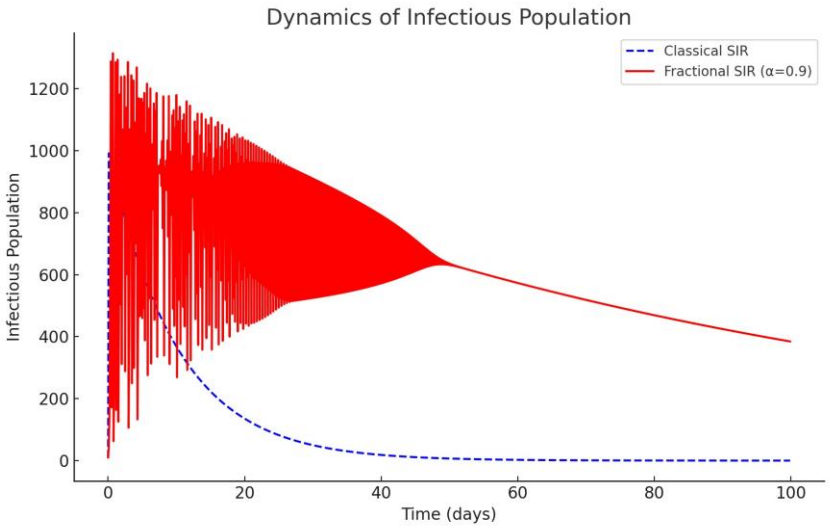

Figure 2: Dynamics of the infectious population in classical and fractional SIR models.

Modeling Long-Term Epidemic Dynamics

The fractional SIR model is particularly suited for diseases with prolonged incubation or recovery periods, such as:

- **Tuberculosis (TB):** Long latent and infectious periods make classical models insufficient. The fractional SIR model accounts for the delayed effects of infection and treatment.
- **COVID-19:** Memory effects are critical for modeling waning immunity and the effectiveness of vaccines over time.
- **Chronic diseases:** Conditions like hepatitis or HIV, where disease progression

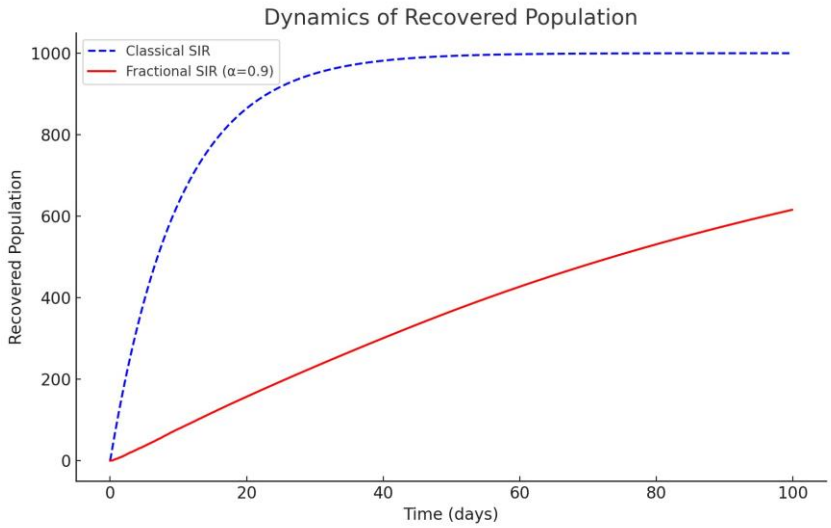

Figure 3: Dynamics of the recovered population in classical and fractional SIR models.

depends on long-term biological history.

The ability to incorporate history-dependent transmission and recovery rates provides better predictions and allows for more targeted interventions.

Designing Public Health Interventions

The proposed model aids in designing more effective public health strategies by accounting for memory effects. For example:

- **Quarantine and Isolation Policies:** Simulations can help determine optimal durations of isolation by analyzing delayed transmission dynamics.
- **Social Distancing Measures:** Understanding how past contacts contribute to current transmission enables better timing and intensity of distancing measures.
- **Testing and Tracing Programs:** Fractional models improve the evaluation of the long-term impact of delayed testing and incomplete tracing programs.

Evaluating Vaccination Strategies

The inclusion of memory effects makes the fractional SIR model ideal for assessing vaccine efficacy in the presence of:

- **Waning Immunity:** The model can simulate how immunity levels decrease over time and affect population-level protection.
- **Booster Doses:** Fractional dynamics allow analysis of booster dose schedules and their long-term impact on epidemic control.
- **Herd Immunity Thresholds:** Memory effects can shift the thresholds for achieving herd immunity, especially when immunity fades or is incomplete.

Optimizing Resource Allocation

Healthcare systems often face resource constraints during epidemics. The fractional SIR model helps optimize resource allocation by:

- **Predicting Epidemic Duration:** Longer epidemic durations due to memory effects can guide stockpiling of medical supplies.
- **Hospital Bed Planning:** Understanding delayed recovery dynamics enables better prediction of hospitalization peaks.
- **Vaccination Rollouts:** Prioritizing high-risk populations based on memory-driven disease dynamics.

Comparative Analysis with Classical Models

The fractional model provides insights not accessible through classical models:

- **Delayed Epidemic Peaks:** Memory effects explain observed delays in real-world epidemics.
- **Prolonged Recovery Times:** Classical models fail to capture the extended duration of recovery for certain diseases.
- **Residual Transmission Effects:** Even after control measures are implemented, memory effects can sustain low-level transmission, which the fractional model accounts for.

The proposed framework expands the toolkit available for epidemiologists and public health planners. Its ability to incorporate history-dependent dynamics makes it a valuable complement to classical models. By capturing the nuanced effects of memory and nonlocal interactions, the fractional SIR model enhances our ability to predict, control, and ultimately mitigate the impact of infectious diseases on society.

Conclusion and Future Work

This paper introduced a novel framework for modeling epidemic dynamics using a fractional SIR model based on Modified Fractional Bell Polynomials. By incorporating memory effects and nonlocal interactions, the proposed framework enhances our ability to analyze and predict the spread of infectious diseases.

Summary of Findings

The key contributions of this work are summarized as follows:

- Developed a fractional SIR model using the Atangana-Baleanu fractional derivative, which effectively captures memory effects in disease transmission and recovery.
- Proposed the Modified Fractional Bell Polynomials to represent solutions of fractional differential equations, providing a powerful tool for analyzing complex dynamics.
- Established theoretical results, including solution representations and consistency with classical models.
- Demonstrated the applicability of the framework through numerical simulations, showing its superiority in capturing delayed dynamics and prolonged recovery.
- Highlighted practical applications in public health, including vaccination strategies, resource allocation, and intervention design.

Future Work

While the proposed framework provides significant insights, several avenues for future research remain:

- **Incorporation of Stochastic Effects:** Real-world epidemics often involve random fluctuations. Extending the model to include stochastic fractional derivatives could enhance its applicability.
- **Network-Based Models:** Applying the framework to contact networks or spatially distributed populations would provide a more detailed understanding of disease spread.
- **Parameter Estimation Techniques:** Developing efficient methods for estimating fractional order α and other parameters from real-world data is critical for practical applications.

- **Comparison with Other Fractional Operators:** Investigating the impact of different fractional derivatives on the model's performance could provide further insights into memory effects.

- **Integration with Machine Learning:** Combining fractional models with machine learning approaches could enhance predictions and provide real-time insights for decision-making.

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